SYMPLECTIC STRUCTURES ON THE COTANGENT BUNDLES OF OPEN 4-MANIFOLDS

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Abstract. We show that, for any two orientable smooth open 4-manifolds X_0, X_1 which are homeomorphic, their cotangent bundles T^*X_0, T^*X_1 are symplectomorphic with their canonical symplectic structure. In particular, for any smooth manifold R homeomorphic to $\mathbb{R}^4$, the standard Stein structure on T^*R is Stein homotopic to the standard Stein structure on $T^*\mathbb{R}^4 = \mathbb{R}^8$. We use this to show that any exotic $\mathbb{R}^4$ embeds in the standard symplectic $\mathbb{R}^8$ as a Lagrangian submanifold. As a corollary, we show that $\mathbb{R}^8$ has uncountably many smoothly distinct foliations by Lagrangian $\mathbb{R}^4$s with their standard smooth structure.

1. Introduction

We begin with some basics, which can be found in [10]. Throughout, assume that all manifolds are orientable, smoothable, and come with a fixed smooth structure unless otherwise indicated. Let N be a smooth manifold of real dimension n. The cotangent bundle of N, T^*N, carries a *canonical 1-form* λ_0 defined, in local coordinates $x_1, \ldots, x_n, y_1 = dx_1, \ldots, y_n = dx_n$, by $\lambda_0 = \sum_{i=1}^n y_i dx_i$. The 1-form $\lambda_0 \in \Omega^1(T^*N)$ is uniquely characterized by the property that $\sigma^*\lambda = \sigma$ for any 1-form σ on N thought of as a section of $T^*N \to N$. Then, for any diffeomorphism $\daleth : M \to N$, $\lambda_0^N = \daleth^{**}\lambda_0^M$ where $\daleth^{**}$ is the induced map $\daleth^{**} : T^*(T^*M) \to T^*(T^*N)$.

From the canonical 1-form we obtain a *canonical symplectic form* $\omega_0 = -d\lambda_0$ on T^*N. In local coordinates, ω_0 has the form $\omega_0 = \sum_{i=1}^n dx_i \wedge dy_i$. As ω_0 depends only on λ_0, any two diffeomorphic manifolds have symplectomorphic cotangent bundles. This fact is the basis of an idea of V. I. Arnol'd which states that the smooth topology of a manifold should be reflected in the symplectic topology of its cotangent bundle.

As a realization of this idea, M. Abouzaid showed in [1] that in dimension $n \equiv 1$ mod 4, an exotic n-sphere S which does not bound a paralizable manifold does not embed as a Lagrangian submanifold of T^*S^n with the the standard symplectic structure; hence the cotangent bundle of such an exotic sphere cannot be symplectomorphic to T^*S^n. As T^*S and T^*S^n are diffeomorphic, the canonical symplectic structure on T^*S can be considered an exotic symplectic structure on T^*S^n.

The existence and classification of exotic symplectic structures on a given smooth manifold is of independent interest. In [14] P. Seidel and I. Smith and in [2] M. Abouzaid and P. Seidel construct exotic symplectic structures on $\mathbb{R}^{2n}$ for $n \geqslant 4$. Also, in [11], M. McLean constructs exotic symplectic structures on T^*S^n for $n \geqslant$

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3. However, these constructions do not arise as cotangent bundles, nor are they symplectomorphic to them in any obvious way.

When $n \neq 4$, smoothing theory tells us that there is a unique smooth structure on the topological manifold $\mathbb{R}^n$ up to homeomorphism. (See [15] for $n > 4$, [12] for $n = 3$.) So among the symplectic structures on $T^*\mathbb{R}^n \cong \mathbb{R}^{2n}$, there is a fixed one which corresponds to the canonical structure on $T^*\mathbb{R}^n$. However in dimension $n = 4$, the topological manifold $\mathbb{R}^4$ admits uncountably infinitely many inequivalent smoothings. For each smooth manifold R, homeomorphic to $\mathbb{R}^4$, T^*R and $T^*\mathbb{R}^4$ are diffeomorphic.

It is then natural to ask: Are any of the exotic symplectic structures on $\mathbb{R}^8$ symplectomorphic to the canonical structure on T^*R for some smooth R homeomorphic to $\mathbb{R}^4$? We answer a stronger question in the negative.

Theorem 1. *Let X_0, X_1 be smooth, open, homeomorphic 4-manifolds. If $\pi_1(X_i) \neq 0$ we assume that there is an s-cobordism between X_0 and X_1.* [1] *Then each of T^*X_0 and T^*X_1 have a fixed Stein structure up to Stein homotopy and the Stein structures on T^*X_0 and T^*X_1 are Stein homotopic. Therefore, T^*X_0 and T^*X_1 are symplectomorphic with their canonical symplectic structures.*

We reserve the definition of a Stein manifold until the next section. A special case of Theorem 1 is the following:

Corollary 2. *Let R be a smooth 4-manifold homeomorphic to $\mathbb{R}^4$. Then the Stein structure on T^*R is Stein homotopic to the standard Stein structure on $\mathbb{R}^8$; hence T^*R and $\mathbb{R}^8$ are symplectomorphic.*

A Lagrangian submanifold L of a symplectic manifold (V, ω) is a submanifold of maximal dimension where $\omega|_L \equiv 0$. In a cotangent bundle, graphs of closed 1-forms are Lagrangian.

Corollary 3. *Let R be any exotic $\mathbb{R}^4$. Then R embeds in the standard symplectic $(\mathbb{R}^8, \omega)$ as a Lagrangian submanifold.*

Proof. The zero section of T^*R is a Lagrangian copy of R which sits inside $\mathbb{R}^8$ as its image under the symplectomorphism. □

Currently, there are no known smooth manifolds which

(1) are homeomorphic to $\mathbb{R}^4$,
(2) have finite handlebody decompositions, and
(3) which are known to be not diffeomorphic to the standard $\mathbb{R}^4$.

If the smooth 4-dimensional Poincaré conjecture is false, such an object exists and arises by puncturing an exotic 4-sphere. Potential examples arise via Gluck twists [7] or from proposed counterexamples to the Andrews-Curtis conjecture. Recall that the Andrews-Curtis conjecture states that balanced presentations of the trivial group can be trivialized using the Andrews-Curtis moves; a collection of moves on group presentations related to elementary Morse moves of 1 and 2 handles. (See Remark 5.1.11 of [8] for examples.) Since no such finite handlebody is currently

[1] Recall that an s-cobordism is an h-cobordism with vanishing Whitehead torsion. In dimension 4, Freedman proved that in the case of "good" fundamental group, such an s-cobordism has a topological product structure. See Theorem 7.1A of [6]. We will only require the existence of such an s-cobordism, not a product structure on it, so we can remove the qualification on π_1.

known to be exotic, all known examples involve highly complicated behavior at infinity.

This complicated behavior at infinity obstructs the usual definitions of Lagrangian Floer homology for non-compact Lagrangians. We then ask the question: Can a Lagrangian Floer homology for be defined for Lagrangians such as we have describe? If so, can any of the above Lagrangian exotic $\mathbb{R}^4$s be distinguished by their Floer homologies?

A symplectic manifold (V, ω) is called *exact* if $\omega = d\alpha$ for some 1-form α. (In the case of the canonical structure on a cotangent bundle $\alpha = -\lambda_0$.) A Lagrangian submanifold L of an exact symplectic manifold is exact if $\alpha|_L$ is exact. Note that $\alpha|_L$ is closed on L as $d\alpha|_L = \omega|_L \equiv 0$. When L has $H^1(L;\mathbb{R}) = 0$, every closed form is exact.

The usual version of the *nearby Lagrangian conjecture* states that if a closed manifold L is an exact Lagrangian in T^*N (with N compact) then L is Hamiltonian isotopic to the zero section of T^*N. We have then shown that the corresponding non-compact version (when N is 4-dimensional and open) is false – without some sort of control at infinity – since such a Hamiltonian isotopy would give a diffeomorphism between any two smooth structures on N.

Let $(X_0, \mathcal{F}_0)$ and $(X_1, \mathcal{F}_1)$ be two smooth manifolds with smooth foliations. We will call these two foliations *smoothly equivalent* if there exists a diffeomorphism $\phi : X_0 \to X_1$ such that $\phi_*(\mathcal{F}_0) = \mathcal{F}_1$. Suppose that a foliated manifold $(X, \mathcal{F})$ admits a *smooth global slice* by which we will mean a smooth manifold Y and smooth embedding $Y \to X$ which intersects every leaf of $\mathcal{F}$ once transversely. It is straightforward to show that any two global slices of $\mathcal{F}$ are diffeomorphic; hence the leaf space of $\mathcal{F}$ is naturally identified with Y with its smooth structure.

Corollary 4. *The standard symplectic $\mathbb{R}^8$ admits uncountably infinitely many smoothly distinct foliations by Lagrangian $\mathbb{R}^4$'s with the standard smooth structure.*

Proof. For each exotic $\mathbb{R}^4$, R, T^*R has a codimension 4 foliation by the fibers of the projection $T^*R \to R$, each a Lagrangian $\mathbb{R}^4$ with its standard smooth structure. Since the leaf space $\mathcal{L}$ is naturally identified with R, $\mathcal{L}$ is a smooth manifold and its smooth type is an invariant of the foliation. The result follows by the uncountability of smooth structures on $\mathbb{R}^4$ together with Corollary 2. □

2. Stein Manifolds

A smooth manifold V of real dimension $2n$, equipped with an almost complex structure J is said to be Stein if J is integrable and V admits a proper holomorphic embedding into $\mathbb{C}^N$ for some N. By [9, 3, 13], a complex manifold (V, J) is Stein if and only if it admits a smooth function $\phi : V \to \mathbb{R}$ which is

(1) proper and bounded below (exhausting) and
(2) is J-convex in the sense that $-dd^{\mathbb{C}}\phi(v, Jv) > 0$ for all v. Here $d^{\mathbb{C}}\phi$ denotes $d\phi \circ J$.

We call the triple (V, J, ϕ) a Stein structure on V. Note that $-dd^{\mathbb{C}}\phi$ is a symplectic form on V compatible with J. In fact, the existence of a Stein structure only requires a weaker condition, due to the following theorem of Eliashberg:

Theorem 5 (Eliashberg [5])**.** *A smooth manifold V^{2n} with $2n > 4$ admits a Stein structure if and only if it admits an almost complex structure J and an exhausting*

Morse function ϕ with critical points of index $\leqslant n$. More precisely, J is homotopic through almost complex structures to a complex structure J' such that ϕ is J' convex.

Associated to every symplectic manifold (V, ω) there is a contractible space of almost complex structures J which are compatible with ω in the sense that $g(v, w) = \omega(v, Jw)$ is a Riemannian metric and $\omega(Jv, Jw) = \omega(v, w)$. When $(V, \omega) = (T^*N, \omega_0)$, we can construct a contractible subspace of these structures explicitly in terms of Riemannian metrics on N. That is, pick a Riemannian metric g_N on N and define J_g in local coordinates $x_1, \dots, x_n, y_1, \dots, y_n$ (for T^*N) by

$$J_g\left(\frac{\partial}{\partial x_i}\right) = \sum_{j=1}^{n} (g_N)_{ij} \frac{\partial}{\partial y_j} \text{ and } J_g\left(\frac{\partial}{\partial y_i}\right) = \sum_{j=1}^{n} -(g_N)^{ij} \frac{\partial}{\partial x_j}$$

Then J_g is compatible with ω_0 and, on the zero section Z, $g|_Z \equiv g_N$. This almost complex structure is not, in general, integrable.

Now, pick an exhausting Morse function $f : N \to \mathbb{R}$. Let $\phi : T^*N \to \mathbb{R}$ be defined by $\phi(x, y) = f(x) + \frac{1}{2}\|y\|^2$. Then ϕ is again an exhausting Morse function, now for T^*N, whose critical points occur along the zero section and have an index-preserving bijection with those of f. As long as the dimension of N is $n > 2$, the conditions of Theorem 5 are satisfied and T^*N admits a Stein structure.

An exhausting Morse function on V^{2n} is called *subcritical* if it has only critical points of index $< n$. A Stein structure (ϕ, J) is called *subcritical* if ϕ is subcritical. In the subcritical case, Y. Eliashberg and K. Cieliebak showed that Stein structures are unique up to homotopy:

Theorem 6 (Eliashberg, Cieliebak [4])**.** *Let $n > 3$ and let (ϕ_0, J_0), (ϕ_1, J_1) be two subcritical Stein structures on V^{2n}. If J_0 and J_1 are homotopic as almost complex structures, then (ϕ_0, J_0) and (ϕ_1, J_1) are homotopic as Stein structures.*

Here a Stein homotopy consists of a concatenation of "simple Morse homotopies" i.e. sequences of Morse birth-deaths and handle slides. In this case, critical points of the ϕ_t do not escape to infinity and Moser's trick[2] applies to give us a 1-parameter family of diffeomorphisms taking one Stein structure to the other. See [for details. Consequently, the underlying symplectic manifolds for the Stein structures (ϕ_0, J_0) and (ϕ_1, J_1) are symplectomorphic.

Before we begin with the proof of Theorem 1. We show the following:

Lemma 7. *If two 4-manifolds X_0, X_1 are homeomorphic, then their cotangent bundles are diffeomorphic as 8-manifolds. If $\pi_1(X_i) \neq 0$, we assume that the X_i are s-cobordant.*

Proof. Let W be an h-cobordism between X_0 and X_1. If $\pi_1(X_i) \neq 0$, then assume that the Whitehead torsion of W vanishes so that W is an s-cobordism. There is a rank 4 real vector bundle T on W which restricts to T^*X_0 and T^*X_1 on ∂W. The bundle T can be obtained by pulling back T^*X_0 via the homotopy equivalence $W \to X_0$. Then we see that $T|_{X_1}$ is isomorphic to T^*X_1 by noting that it has the requisite characteristic classes.

The unit sphere bundle $S(T)$ is then an s-cobordism between 7-manifolds – the unit sphere bundles of T^*X_i – and, by the s-cobordism theorem, a product.

Now, taking the unit disc bundle $D(T)$, we get an s-cobordism of 8-manifolds with boundary. As we have seen, it is a product on the boundary. As this is again

[2]See Section 3.2 of [10] for the compact case.

a product by the s-cobordism theorem, the diffeomorphism of the T^*X_i follows by restricting to the interior. □

Proof of Theorem 1. If X_0 and X_1 are homeomorphic, then T^*X_0 and T^*X_1 are diffeomorphic by Lemma 7. Choose some representative V of this diffeomorphism type and particular diffeomorphisms $\daleth_i : V \to T^*X_i$. The canonical 1-forms, symplectic forms and choices of almost complex structures for the cotangent bundles pull back to λ_i, ω_i and J_i on V.

First, we show that J_0 and J_1 are homotopic as almost complex structures. Write $\mathcal{I}(n)$ for the space $GL^+(2n,\mathbb{R})/GL(n,\mathbb{C})$ which classifies almost complex structures on $\mathbb{R}^{2n}$. The space $\mathcal{I}(n)$ is homotopy equivalent to $SO(2n)/U(n)$ and, when $n=4$, we can compute several of the homotopy groups:

$$\begin{aligned} &\pi_0\mathcal{I}(4)=0, \quad \pi_1\mathcal{I}(4)=0, \quad \pi_2\mathcal{I}(4)=\mathbb{Z},\\ &\pi_3\mathcal{I}(4)=0, \quad \pi_4\mathcal{I}(4)=0, \quad \pi_5\mathcal{I}(4)=0,\\ &\pi_6\mathcal{I}(4)=\mathbb{Z}, \end{aligned}$$

The obstructions to a homotopy between J_0 and J_1 lie in $H^i(V,V^{(i-1)};\pi_i\mathcal{I}(4))$ where $V^{(i-1)}$ is the $i-1$–skeleton of V. As $H^i(V;\mathbb{Z})=0$ for $i\geqslant 4$, the only non-trivial group in this list is $H^2(V,V^{(1)};\pi_2\mathcal{I}(4))\cong H^2(V,V^{(1)};\mathbb{Z})\cong H^2(X_i,X_i^{(1)})$. However, J_0 and J_1 are homotopic over the 2-skeleton so this obstruction vanishes. (Clearly they both have the same first Chern class.) Therefore, J_0 and J_1 are homotopic as almost complex structures.

As we saw above, for each Morse function f_i on X_i, we obtain Morse functions ϕ_i on T^*X_i whose critical points (and, once a metric is chosen, flow lines along the zero section) can be identified with those of f_i. Note that the ϕ_i will be proper and bounded below when the f_i are.

In order to apply Theorem 6, we will need to show that the X_i admit Morse functions without index 4-critical points. To construct such a Morse function, begin with a Morse function which is bounded below and whose critical values are discrete, with a single critical point per critical value. With such a choice, X_i is identified with a composition of elementary cobordisms.

Let x_4 be an index 4 critical point. Then the unstable manifold of x_4 must include trajectories to at most finitely many index 3 critical points. Further, if x_3 is an index 3 critical point, its stable manifold must meet at most 2 index 4 critical points. (Since the stable manifold is 1 dimensional.) As $H_4(X_i;\mathbb{Z})=0$, the boundary map $\partial : C_4(X)\to C_3(X)$ is injective; so, after possibly performing some handle-slides, we can cancel any index 4 critical point with an index 3 critical point. Therefore, each of the X_i admit a Morse function f_i without index 4-critical points.

Let (ϕ_i, J_i) be the Stein structures on V constructed using Theorem 5 where J_i is a complex structure homotopic to the almost complex structure J_i' constructed from the metric on X_i. As each of the ϕ_i are subcritical, we then apply Theorem 6 to see that the (ϕ_i, J_i) are Stein homotopic. □

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